Predicting chaos with second method of Lyapunov

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Abstract: We overview several analytic methods of predicting the emergence of chaotic motion in nonlinear oscillatory systems. A special attention is given to the second method of Lyapunov, a technique that has been widely used in the analysis of stability of motion in the theory of dynamical systems but received little attention in the context of chaotic systems analysis. We show that the method allows formulating a necessary condition for the appearance of chaos in nonlinear systems. In other terms, it provides an analytic estimate of an area in the space of control parameters where the largest Lyapunov exponent is strictly negative. A complementary area thus comprises the values of controls, where the exponent can take positive values, and hence the motion can become chaotic. Contrary to other commonly used methods based on perturbation analysis, such as e.g., Melnikov criterion, harmonic balance, or averaging, our approach demonstrates superior performance at large values of the parameters of dissipation and nonlinearity. Several classical examples including mathematical pendulum, Duffing oscillator, and a system of two coupled oscillators, are analyzed in detail demonstrating advantages of the proposed method compared to other existing techniques.

Keywords: direct method of Lyapunov, Lyapunov exponents, Melnikov method, harmonic balance, averaging, saddle-node bifurcation, local expansion rates, pendulum, Duffing

1. Introduction

One of the classical problems in the analysis of nonlinear systems is predicting the properties of motion from the functional form of the governing differential equations. In particular, the arising of various instabilities leading to bifurcations of regular motions and, finally to the appearance of strange attractors in the phase space, have been at the focus of active research in the last decades [1,2]. In general terms, a typical problem posing of this type looks like the following one.

Given a dynamical system described by a set of differential equations

$$\frac{dx}{dt} = F(x, t, c), \quad x \in \mathbb{R}^n; c \in \mathbb{R}^m$$

(1)

where \(c\) is vector of \(m\) control parameters, specify the areas in the \(c\)-space where the behavior is periodic, quasiperiodic, or chaotic. The borders of such
areas in a two-dimensional cross-section of the $e$-space present bifurcation lines, and the cross-sections themselves constitute state diagrams. Despite considerable efforts aimed at designing analytic techniques for predicting different bifurcations and conditions for the transition to chaos in dynamical systems, the number of tools available for this kind of analysis remains limited, and each method deals with a very restricted class of differential equations.

In comparatively simple cases like, e.g., saddle-node bifurcation in periodically forced systems or Andronov-Hopf bifurcation in autonomous oscillators, the bifurcation conditions can be found by a two-step procedure, when an approximation to a solution (typically, periodic or quasiperiodic) is found first, followed by the stability analysis. The bifurcation lines in this approach are the borders of stability areas found at the final step of analysis.

Several approaches have been proposed for finding the conditions for the transition to chaos. The methods are known by the names of their inventors, the most famous ones being those of Chirikov, Melnikov, and Shilnikov. It should be noted, however, that all of these methods can be called “indirect” ones, since they do not find the conditions of chaotization of motion, but analyze the phenomena accompanying chaos. In particular, in the case of Hamiltonian systems treated by Chirikov method, the analysis predicts the phenomenon of overlap of resonances, Melnikov method finds the position of homoclinic bifurcation (intersection of stable and unstable manifolds of a saddle state), whereas Shilnikov method looks for special trajectories, saddle-focus loops, prerequisite of a chaotic motion.

On the other hand, it appears natural to use the very definition of chaotic motion, i.e. exponential instability of chaotic trajectories with respect to variations in initial conditions, for predicting its appearance. The sensitivity to initial conditions is described by the spectrum of Lyapunov exponents $\lambda_i$ that characterize average growth rates of perturbations in orthogonal directions along a trajectory. It has been recently demonstrated [3] that a necessary condition for chaos can be efficiently formulated in terms of Temporally Local Lyapunov Exponents (sometimes also called Instantaneous Lyapunov Exponents, ILE), $\mu_i$, that represent the local time-dependent expansion rates at a given point on the trajectory. The standard Lyapunov exponents $\lambda_i$ are simply long time averages of corresponding ILE. Requirement for all ILE to be strictly negative is equivalent to imposing a condition of asymptotic stability on the fiducial trajectory. In this paper, we show that analysis in terms of ILE is equivalent to that of direct (second) method of Lyapunov. Since chaos usually appears in dynamical systems as a result of a loss of stability by some periodic orbits, its prediction requires using a variant of the Lyapunov method developed for non-autonomous systems [4]. The conclusion on the instability of motion that can lead to chaos can be derived from very similar arguments to those formulated in Chetaev theorem [4, 5].
2. Lyapunov exponents and direct (second) method

Lyapunov exponents of an arbitrary system of type (1), are usually calculated by the analysis of solutions in linearized system

$$\frac{dy}{dt} = J(x^*(t))y,$$

where $J(x^*(t)) = \frac{\partial (x(t))}{\partial x}$ is n×n time dependent Jacobian matrix, y is an n-vector of perturbations around the trajectory $x^*(t)$. The standard algorithm of calculating the spectrum of Lyapunov exponents [6] consists in solving the equations (2) simultaneously with (1) for a set of mutually orthonormal vectors $\{ y_k \}, \ (k = 1, 2, ..., n)$ and estimating expansion rates (ILE)

$$\mu_k(t) = \frac{d[\ln \rho_k(t)]}{dt}$$

for the lengths $\rho_k = ||y_k||$ of the vectors $\{ y_k \}$. Then, estimates of Lyapunov exponents can then be derived as time averages

$$\lambda_k = \lim_{T \to \infty} \frac{1}{T} \int_0^T \mu_k(t) dt$$

Stability of the solution $x^*(t)$ is thus defined by the time-dependent function $\mu_k(t)$, i.e. it is asymptotically stable if $\mu_k(t) < 0$ for all times, and hence the largest Lyapunov exponent is negative. As it is discussed in [3, 4], the conditions of stability can be often derived by analysing the properties of Eqs.(1-2), without finding their exact solutions. In particular, it had been demonstrated that the condition for stability in a wide class of nonlinear oscillatory systems is defined by the amplitude of motion only, i.e. the motion becomes unstable when the trajectory crosses certain threshold. The estimate of the critical amplitude can be obtained by some other technique and formulate the stability (instability) criteria in terms of control parameters.

On the other hand, the stability of motion can also be analysed by the direct method of Lyapunov [7], which establishes the stability conditions from the analysis of so-called Lyapunov function, $V(y)$. In order to be stable, a solution $x^*(t)$ should evolve in a way that ensures that the function $V(y)$ always keeps different sign compared to its time derivative, $V'(y)$. Selecting, for example, the Lyapunov function to be definite positive, one can predict the stability of solutions in the areas of the phase space and the space of controls $c$ where the condition $V'(y) < 0$ is satisfied.

We are now in a position of demonstrating equivalence of the stability analysis in terms of ILE and direct method of Lyapunov. Indeed, selecting the Lyapunov function in the form that guarantees its positive definiteness...
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\[ V(y) = \sum_{i=1}^{n} y_i^2, \quad (5) \]

one can see that it coincides with the squared norm \( \rho_i = \|y_i\| \) of an arbitrary vector from the set \( \{y_k\} \). Then considering the ratio

\[ \frac{\dot{V}}{V} = \frac{d}{dt} \frac{\rho_i^2}{\rho_i^2} = 2 \frac{\rho_i}{\rho_i} = 2 \mu_i \quad (6) \]

we conclude that the demand for the opposite signs of the functions \( V(y) \) and \( \dot{V}(y) \) is equivalent to the requirement of \( \mu_i(t) < 0 \). Therefore, the problem of stability is reduced to the analysis of sign of the function \( \dot{V}(y) \), and the problem consists in finding the areas in the space of controls where the condition \( \dot{V}(y) < 0 \) is satisfied in certain range of variation for the variable

\[ x : x_{\text{min}} < x^*(t) < x_{\text{max}}. \]

The interval of values \( [x_{\text{min}}; x_{\text{max}}] \) that locates the solution \( x^*(t) \) can be estimated by other methods, such as e.g., averaging, harmonic balance, or multiple time scale analysis [8]. As it is shown in [3], the demand of \( \dot{V}(y) < 0 \) imposed by Lyapunov method leads to the necessity of introducing a linear coordinate transform that converts the function (5) to that of Lyapunov-type and hence makes feasible the stability analysis. In the next chapter, we demonstrate the efficiency of the method with several examples of classical nonlinear oscillators.

3. Examples of nonlinear oscillators

3.1. Single-well cubic potential

Differential equation describing the time evolution of Duffing oscillator with nonlinearity of hardening type \( (\varepsilon > 0) \) reads as

\[ \frac{d^2x}{dt^2} + \delta \frac{dx}{dt} + \omega_0^2 x + \varepsilon x^3 = e(t) \quad (7) \]

It is known that this system has chaotic solutions, even in the simplest case of one-frequency excitation \( e(t) = F \cos(\omega t) \), at certain combination of the parameters \( F \) and \( \omega \) [9]. For hardening type of nonlinearity, no analytic methods exist for predicting the appearance of chaotic motion. To apply Lyapunov stability analysis, one needs to consider a notation \( (x; \dot{x}) \rightarrow (x_1; x_2) \) and a linear coordinate transform [3]

\[ z_1 = \frac{\delta}{\delta + 2\omega_0} x_1; \quad z_2 = x_1 + \frac{\delta}{2} x_2 \quad (8) \]

that leads to the following expression for the ratio \( \dot{V}/V \).
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\[
\frac{\dot{V}}{V} = -\delta + \left( \frac{6\beta x^2}{\delta + 2\omega_0} \right) \sin \Phi
\]

and allows estimating maximal stable amplitude as

\[
x_{\text{max}} = \sqrt{\frac{\delta(\delta + 2\omega_0)}{3\varepsilon}}
\]

If the value of \( x \) exceeds the threshold given by Eq.(10) the motion of the system may (although not necessarily) become unstable. In order to estimate the amplitude of motion we use the simplest approach by calculating the response function of linear oscillator, i.e. Eq. (7) at \( \varepsilon = 0 \)

\[
x_{\text{max}} = F \sqrt{\left( \omega_0^2 - \omega^2 \right) + \delta^2 \omega^4}
\]

Combining Eqs. (10-11), we obtain the threshold line in the parameter plane \( F - \omega \). In Fig. 1 we plot the result, together with saddle-node bifurcation lines obtained with averaging method [8]. As one can see, the combination of the proposed criterion with linear response theory works well, accurately predicting the onset of saddle-node bifurcation, i.e. the lowest order instability with respect to the parameter \( F \). Other bifurcations, including period-doubling cascades and chaos, occur at much higher values of \( F \).

![Fig.1. Lyapunov stability of single–well Duffing-type oscillator with hardening nonlinearity (squares); saddle-node bifurcation (heavy line) in the vicinity of principal resonance \( \omega = \omega_0 \).

Parameters: \( \delta = 0.5; \omega_0 = 1; \varepsilon = 1 \).

3.2. Mathematical pendulum

Equation describing the dynamics of a simple pendulum can be written as

\[
\frac{d^2 x}{dt^2} + \delta \frac{dx}{dt} + \sin(x) = f(t)
\]

Performing the analysis similar to the case of Duffing oscillator, we come to the two cases that should be considered, depending on the value of
dissipation parameter $\delta$

1. If $\delta < 1$, then the maximal stable displacement is defined by
\[
|v_{\text{max}}| = \cos^{-1}\left[1 - (1 - \delta)^2\right]
\]
(13a)

2. If $\delta > 1$, then the size of stability area does not depend on dissipation and is limited by the value of
\[
|v_{\text{max}}| = \frac{\pi}{2}
\]
(13b)

Fig. 2. Phase diagram of the pendulum system (12). The borders of Lyapunov stability areas are shown as heavy lines. They are compared to those obtained with Melnikov criterion (dashed lines) and actual position of chaotic areas obtained by direct numerical integration of Eq. (12) (lite lines). Blue, pink, and green lines correspond to $\delta = 0.25$, 0.5, and 1, respectively.

In order to check the efficiency of the proposed criterion and compare it to the results of previous works [10], we consider the case of harmonic excitation of type $f(t) = F \cos(\omega t)$ and use the following formula for an approximate solution of the Eq.(12)
\[
x = A \cos(\omega t + \phi)
\]
(14)
where the amplitude $A$ is as a root of the equation
\[
2J_1(A) - \omega^2 A^2 + (A \omega \delta)^2 = F^2
\]
(15)
where $J_1(A)$ is Bessel function of the first kind [10]. By substituting the value of maximal stable amplitude (13) to Eq. (15), we obtain the borderline of the asymptotic stability area in explicit form $\gamma(\omega)$. Extensive numerical analysis of the oscillator (12) has been performed in [10] where the loci of chaotic areas on the plane of control parameters $\omega - F$ have been established at the value of $\delta = 0.25$. In Fig.2, we plot the borders of chaotic areas found numerically, Melnikov condition [11] and Lyapunov stability lines Eqs. (13a) and (15) at $\delta = 0.25$, 0.5, and 1. Apparently, the latter
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analysis provides a good prediction for the onset of chaotic oscillations, especially in the low-frequency part of the diagram where Melnikov’s method strongly underestimates the chaos threshold. We also note that the latter provides an incorrect prediction of the position of chaotic area at high value of the dissipation parameter \( \delta = 1 \), whereas our analysis always provides a correct estimate for the threshold amplitude of the external force.

### 3.3. Two coupled oscillators

The last example is a system of two coupled resonance oscillators under the action of harmonic external signal that can be described as

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\delta x_2 - \omega_0^2 x_1 - \omega_1^2 x_1^3 + \kappa (x_3 - x_1) + F \cos(\omega t) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -\delta x_4 - \omega_0^2 x_3 - \omega_3^2 x_3^3 - \kappa (x_3 - x_1)
\end{align*}
\]

Lyapunov analysis leads to the identical estimate of the stability threshold for each of the interacting oscillators

\[
x_{1,2}^{\text{max}} < \sqrt{\frac{\delta \omega_0 \omega_2}{3 \varepsilon (\omega_1 + \omega_2)}} ; \quad \text{where} \quad \omega_{1,2}^2 = \kappa + \omega_0^2 - \frac{\delta^2}{4} + \kappa
\]

Estimate of the amplitudes by, e.g., the method of harmonic balance requires solving the system of four nonlinear algebraic equations [13]. However, using the linear approximation approach described in Section 3.1, i.e. considering only linear terms in (16), we get
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\[ x_1^{\text{max}} = F\sigma \sqrt{\left(\omega_0^2 - \omega^2\right)^2 + (\delta \omega)^2}; x_2^{\text{max}} = kF\sigma \]  

(18)

where

\[ \sigma = 4 \prod_{i=1}^{4} \sqrt{\lambda_i^2 + \omega^2} \quad ; \quad \lambda_{1+4} = -\delta \pm \sqrt{\delta^2 - 4\omega_0^2 \pm \kappa} \]

In Fig. 3, we plot the lines of saddle-node bifurcation [12], together with the threshold line provided by our criterion. One can note a good quality of prediction given by Lyapunov stability analysis combined with the linear response theory. Note that other instabilities, including chaotic motion are also located above the found threshold line [12].

5. Conclusions

The Second (direct) method of Lyapunov is shown to be an efficient technique for predicting the appearance of chaotic instability in nonlinear oscillatory systems. Due to the approximate character of additional methods that have to be used for estimating the maximal amplitude of the response signal, the method may over- or under-estimate the threshold value of controls where chaos occurs. This can be seen, e.g., in the high-frequency parts of Fig.2 where the predicted threshold lines lie well below actual chaotic areas, or in Fig. 3, where the threshold line lies above the saddle-node bifurcations. This means that special care should be taken in applying this criterion in combination with other approximate methods, especially in the systems possessing multiple coexisting attractors. However, compared to other methods like, e.g., Melnikov one, Lyapunov approach has an advantage of formulating a necessary condition for chaos, i.e. finding the maximal stable amplitude of motion, which is important for many applications.

References

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