TECHNOLOGICAL FORECASTING AND SOCIAL CHANGE 27, 39–61 (1985)

Two Generalized Rational Models for Forecasting Innovation Diffusion

CHRISTOS SKIADAS

ABSTRACT

This article demonstrates the development and use of two Generalized Rational Models I and II (GRM I and II) representing innovation diffusion. Specifically, the GRM II covers the same area as the NSRL model, which includes the Coleman and the Blackman/Fisher-Pry models, while the GRM I covers the same area as a modified NSRL model (mod. NSRL), also introduced hereby, and including Floyd, Blackman/Fisher-Pry, Sharif-Kabir and Exponential models. Both the GRM I and the GRM II provide a form of differential equation which always has for a solution a fact which cannot be met when dealing with the NSRL and mod. NSRL models.

Some applications are presented, first to illustrate the wide applicability and the usefulness of the models and second to demonstrate the alternate use of the GRM I and mod. NSRL, and GRM II and NSRL models, which usually approximate very well.

Introduction

In recent years, the literature describing the innovation diffusion process has centered on mathematical models with changing characteristics in the diffusion coefficient b. On the basis of this perspective, models such as the Sharif-Kabir model [13, 14], the Generalized Model (Mahajan and Schoeman) [2, 5, 6, 7, 8] and more recently, the NSRL model [3] have been proposed.

The original sources of these models are the Coleman [15] and the Blackman/Fisher-Pry models (S-curve) [1, 4, 9, 10, 15], described by the following differential equations.

Coleman:

$$\dot{f} = b(F - f), \tag{1}$$

Blackman/Fisher-Pry:

$$\dot{f} = b\,\frac{F - f}{F}\,f, \tag{2}$$

where $\dot{f} = df/dt$ is the rate of adoption, f the number of adopters or the market share of

CHRISTOS SKIADAS is Research Collaborator, Processes and Systems Engineering, in the Department of Chemical Engineering at Athens National Technical University, Athens, Greece.

Address reprint requests to Christos Skiadas, Londou 4, 106 81 Athens, Greece.

 0040-1625/85/$3.30

a product at time t, F the total number of the potential adopters or the upper limit of the market share of a product, and b the diffusion parameter representing the number of adopters or the covered market share per unit time.

An analysis of the Blackman/Fisher-Pry model leads to the following: As $f \rightarrow F$, the Coleman model is approached and as $f \rightarrow 0$, it yields the Exponential model represented by the equation of the Exponential model:

$$\dot{f} = bf. \tag{3}$$

The early stages of the Blackman/Fisher-Pry model follow the Exponential model, which can describe a "pure imitation process." The later stages represent a process that can be described by the Coleman model. The intermediate stages of the Blackman/Fisher-Pry model represent the well-known interaction process between adopters and potential adopters. The three models are represented in Figure 1. The Blackman/Fisher-Pry model has a point of inflection at $f = F/2$.

Mahajan and Schoeman [8] have developed the diffusion parameter in the Coleman model, assuming that b changes over time according to the equation

$$b(t) = a + b\frac{f}{F}, \tag{4}$$

where a and b are parameters. Substituting in eq. (1), they obtained a Generalized model:

$$\dot{f} = \left(a + b\frac{f}{F}\right)(F - f). \tag{5}$$

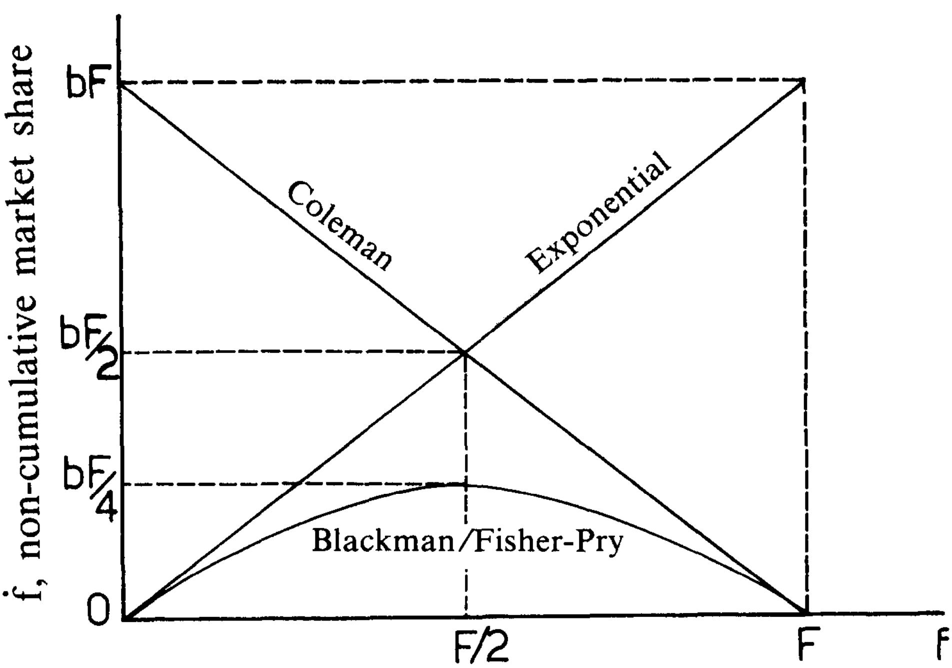

Fig. 1. Comparison between Coleman, Exponential, and Blackman/Fisher-Pry models, for the same b and F.

This last equation can be written as

$$\dot{f} = a(F - f) + b\frac{F - f}{F}f, \tag{6}$$

which is a linear combination of the Coleman and the Blackman/Fisher-Pry models.

Differentiation of eq. (5) with respect to f yields the point of inflection of the Generalized model, for which

$$f = \frac{F}{2}\left(1 - \frac{a}{b}\right). \tag{7}$$

For $0 < a/b < 1$, a point of inflection within the range $0 < f < F/2$ is obtained, and the model takes the form of curve II (Figure 2). The process approaches a Blackman/Fisher-Pry curve when $0 < a/b \ll 1$.

For $a/b > 1$, no point of inflection is reached and the model takes the form of curve I (Figure 2). For $b \ll a$, the process approaches the Coleman model.

The Floyd model is represented by the following differential equation [3]:

$$\dot{f} = b\frac{(F - f)^2}{F^2}f, \tag{8}$$

which shows a point of inflection at $f = F/3$.

The Sharif-Kabir model, giving a point of inflection within the limits of $F/3$ and

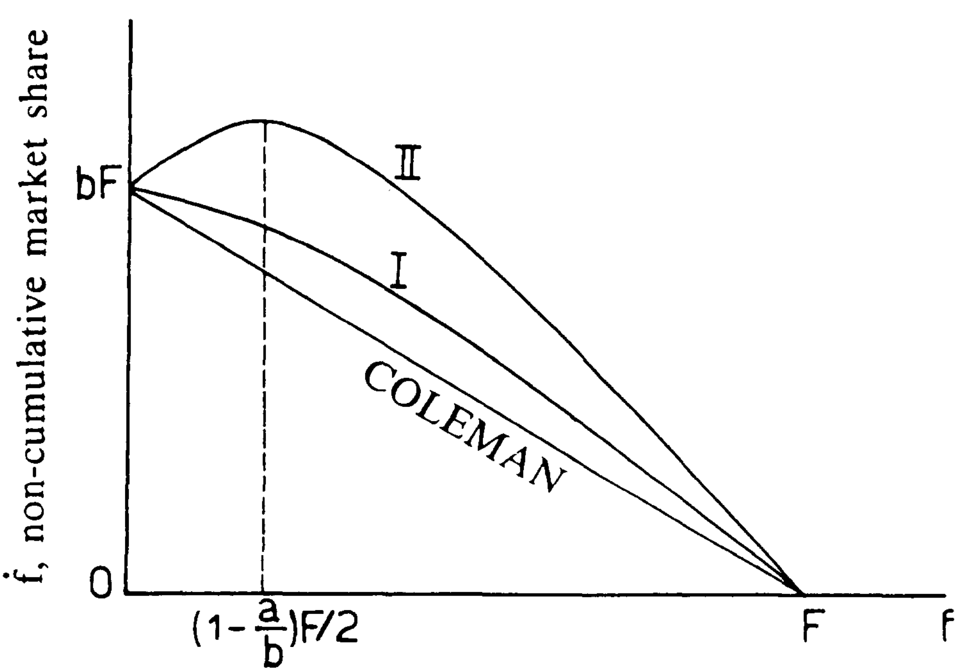

Fig. 2. Generalized Model, curves I and II.

F/2, covers the area between the Blackman/Fisher-Pry and Floyd models (Figure 3). This model is represented by the following differential equation [3, 13]:

$$\dot{f} = \frac{b}{F} \cdot \frac{(F - f)^2}{F - (1 - \sigma)f} f, \tag{9}$$

where $0 \leqslant \sigma \leqslant 1$.

Differentiation of eq. (9) with respect to *f* yields the point of inflection of the Sharif-Kabir model, for which [3]

$$f = F\frac{3 - \sqrt{1 + 8\sigma}}{4(1 - \sigma)} = F\frac{2}{3 + \sqrt{1 + 8\sigma}}. \tag{10}$$

Easingwood, Mahajan and Muller introduced the NSRL model [3], which is represented by the following differential equation:

$$\dot{f} = b\left(\frac{f}{F}\right)^{\delta} (F - f). \tag{11}$$

The NSRL model gives a point of inflection varying between 0 and F as δ changes from 0 to ∞, according to the following equation [3]:

$$f = F\frac{\delta}{1 + \delta}. \tag{12}$$

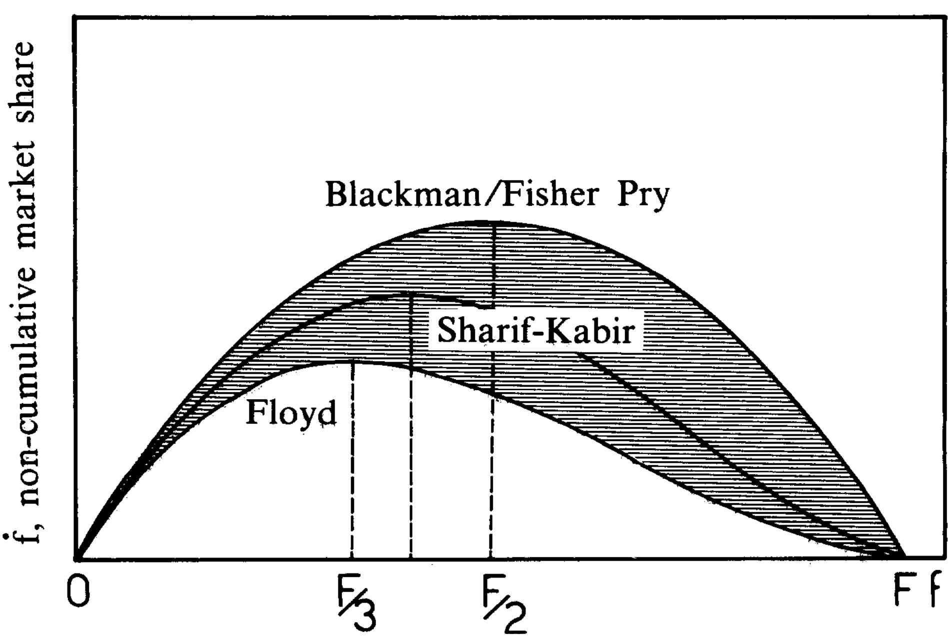

Fig. 3. Comparison between Blackman/Fisher-Pry, Floyd, and Sharif-Kabir models, for the same b and F.

As shown in Figure 4, for $0 < \delta < 1$ the NSRL model covers the region between Coleman and Blackman/Fisher-Pry models. For $\delta = 2$, the NSRL becomes:

$$\dot{f} = \frac{b}{F^2}(F - f)f^2. \tag{13}$$

This model has a point of inflection for $f = 2F/3$,

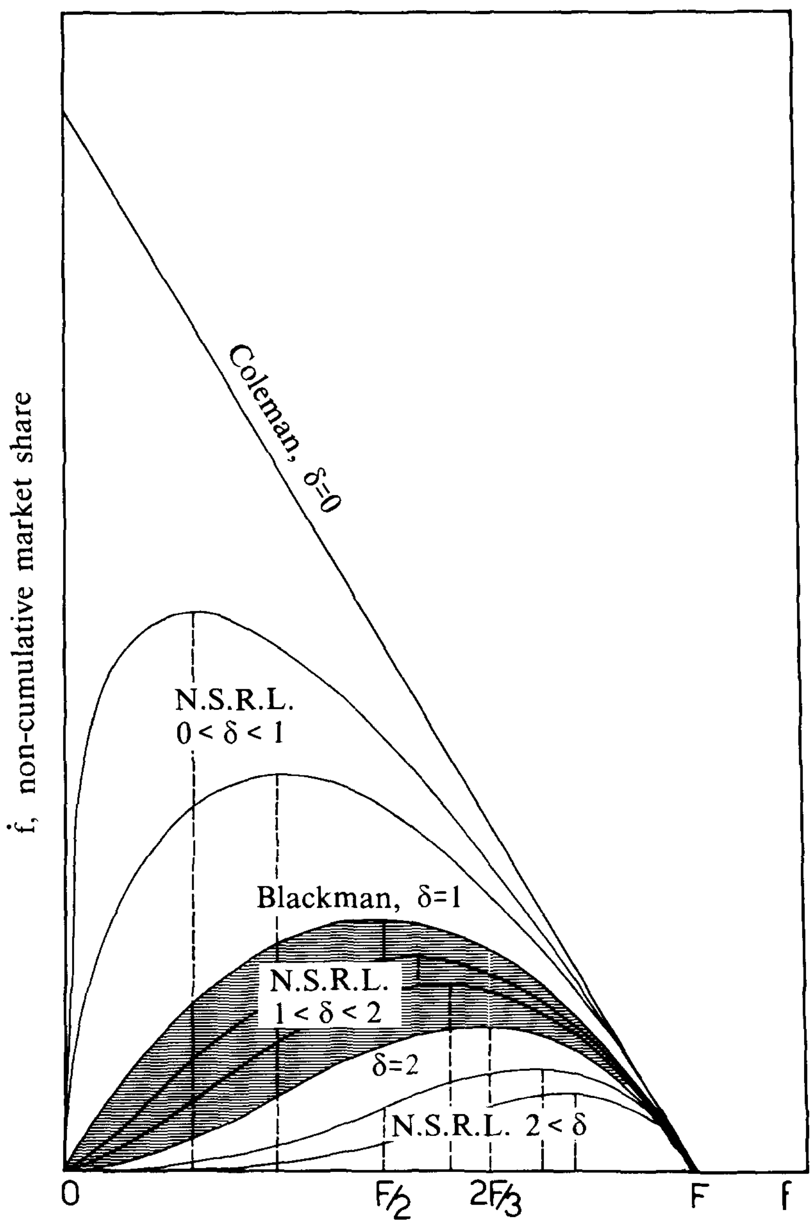

Fig. 4. NSRL model for various values of δ and for the same F and b.

For $1 < \delta < 2$, the models obtained are left skewed, with a point of inflection between $f/2$ and $2F/3$. When $2 < \delta$, the point of inflection varies from $2F/3$ to F (Figure 4).

The NSRL model does not cover the region between the Blackman/Fisher-Pry and Floyd models or that below the Floyd model. Furthermore, the NSRL model usually leads to a form of equation which has a solution only for certain values of δ [3].

The Modified NSRL Model (Mod. NSRL)

The previous analysis of the NSRL model leads to an extension which may cover the region below the Floyd model, between the Floyd and Blackman/Fisher-Pry models, and between the Blackman/Fisher-Pry and Exponential models. By using the Exponential model as a basic model (eq. (3)), and assuming that parameter b changes over the time according to the equation

$$b(t) = b(1 - \frac{f}{F})^{\delta}, \tag{14}$$

substitution of eq. (14) in eq. (3) yields

$$\dot{f} = b(1 - \frac{f}{F})^{\delta} f, \tag{15}$$

which is the differential equation of the model here defined as the "modified NSRL" model (mod. NSRL).

When $\delta = 0$, the mod. NSRL model gives the exponential model; when $\delta = 1$, it gives the Blackman/Fisher-Pry model; when $\delta = 2$, the Floyd model is reached. When $0 < \delta < 1$, the mod. NSRL model covers the region between the exponential and Blackman/Fisher-Pry models. When $\delta < 2$, the region below Floyd model is obtained (Figure 5). Finally, when $1 < \delta < 2$, the region covered by the mod. NSRL model (the shaded area in Figure 5), is identical to that bounded by the Blackman/Fisher-Pry and Floyd models. The same region is also covered by the Sharif-Kabir model.

By differentiating eq. (15) with respect to f, solving the resulting equation for f, by equating it to zero, a point of inflection can be obtained:

$$f = \frac{F}{1 + \delta}, \tag{15a}$$

varying from F to 0 when δ is varying from 0 to ∞.

The modified NSRL model, like the NSRL model, usually leads to a form of equation which has a solution only for certain values of δ.

The Generalized Rational Model I (GRM I)

After the introduction of the mod. NSRL model, our main effort is centered on the construction of a new model, covering the same region as the mod. NSRL and providing a form of differential equation which has always a solution.

For this purpose, eq. (15) representing the mod. NSRL model, is expressed as follows:

$$\dot{f} = b\left(1 - \frac{f}{F}\right)^{n - F} f \tag{16}$$

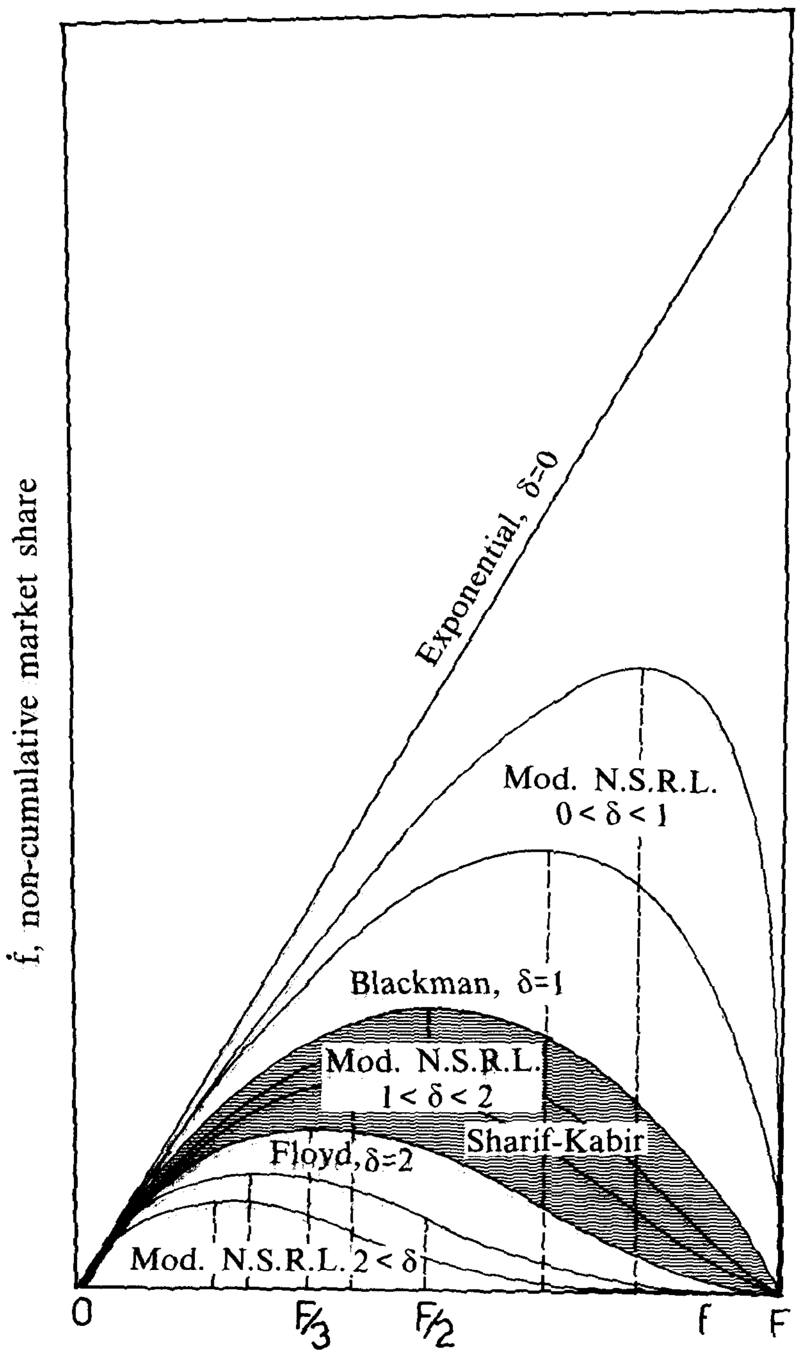

Fig. 5. Modified NSRL model for various values of δ and for the same F and b.

where $n - \varepsilon = \delta$ if $0 \leq \varepsilon < 1$, n is an integer which varies from 1 to ∞ when δ varies from 0 to ∞, and $\delta < n \leq \delta + 1$. From eq. (16) follows

$$\dot{f} = b \frac{\left(1 - \frac{f}{F}\right)^n}{\left(1 - \frac{f}{F}\right)^\varepsilon} f. \tag{17}$$

Substituting the denominator in the right side of eq. (17) by the approximation provided in Appendix A (eq. (A-5)), equation (17) gives

$$\dot{f} = b \frac{\left(1 - \frac{f}{F}\right)^n}{1 - \varepsilon \frac{f}{F}} f. \tag{18}$$

Substituting ε with $(1 - \sigma)$ in eq. (18) with rearrangements yields

$$\dot{f} = b \frac{(F - f)^n}{F^{n-1}[F - (1 - \sigma)f]} f, \tag{19}$$

where $n = 1, 2, \ldots$, and $0 \leq \sigma \leq 1$.

The model expressed by eq. (19) is defined as the "Generalized Rational Model I" (GRM I).

This model covers the same region as the mod. NSRL model illustrated in Figure 5. Setting $n = 1$ and $\sigma = 0$ in eq. (19) yields the Exponential model; setting $n = 1$ and $\sigma = 1$ or $n = 2$ and $\sigma = 0$ yields the Blackman/Fisher-Pry model; and setting $n = 2$ and $\sigma = 1$ yields the Floyd model.

Furthermore, eq. (19), which is a first-order nonlinear differential equation with a complicated form [12], can be transformed in a form that can be easily solved by integration. As derived in Appendix C, the solution of eq. (19) of the GRM I model is provided by eq. (C-4), which is

$$\ln f - \ln(F - f) + \frac{F}{F - f} + \ldots + \frac{F^{n-2}}{(n - 2)(F - f)^{n-2}} + \sigma \frac{F^{n-1}}{(n - 1)(F - f)^{n-1}} = c + bt, \tag{20}$$

where $n = 2, 3, \ldots$. When $n = 1$, the solution is given by eq. (C-6) in Appendix C:

$$\ln f - \sigma \ln(F - f) = c + bt, \tag{21}$$

where c is a constant, the value of which results by setting $t = 0$.

Differentiating eq. (19) with respect to f, solving the resulting equation for f, by equating it to zero, the inflection point of the GRM I model is obtained by

$$f = F\frac{(n + 1) - \sqrt{(n + 1)^2 - 4n(1 - \sigma)}}{2n(1 - \sigma)}$$

or

$$f = \frac{2F}{(n + 1) + \sqrt{(n + 1)^2 - 4n(1 - \sigma)}} \qquad (22)$$

If the mod. NSRL and GRM I models have the same value for f in the point of inflection, then eqs. (15a) and (22) lead to the following equation for parameters σ, n and δ:

$$\sigma = \frac{\delta(1 + \delta - n)}{n}. \qquad (23)$$

Hence,

$$\text{if } n = 1\text{, then } \sigma = \delta^2, \qquad (24)$$

$$\text{if } n = 2 \text{ then } \sigma = \frac{\delta}{2}(\delta - 1). \qquad (25)$$

When $n = 1$, the GRM I_1 model is obtained with a differential equation provided by setting $n = 1$ in eq. (19):

$$\dot{f} = b\frac{(F - f)}{F - (1 - \sigma)f}f. \qquad (26)$$

This model has a point of inflection which yields, by substituting n with 1 in eq. (22):

$$f = F\frac{1 - \sqrt{\sigma}}{1 - \sigma} = \frac{F}{1 + \sqrt{\sigma}}. \qquad (27)$$

The point of inflection varies from $F/2$ to F when σ varies from 1 to 0. The GRM I_1 model covers the region between the Blackman/Fisher-Pry and Exponential models. The same region is covered by the mod. NSRL model when $0 \leqslant \delta \leqslant 1$.

For $n = 2$, eq. (19) provides the differential equation of the GRM I_2 model, which is identical with the Sharif-Kabir model expressed by the differential equation (9) and with the point of inflection given by eq. (10). Setting $n = 2$ in the expansion form (eq. (20)), the equation of the GRM I_2/Sharif-Kabir model yields

$$\ln f - \ln(F - f) + \sigma\frac{F}{F - f} = c + bt. \qquad (28)$$

This model covers the same region covered by the mod. NSRL model when $1 \leqslant \delta \leqslant 2$.

Comparisons between the two models have been made in Figures 6–8, where three pairs of the GRM I_2/Sharif-Kabir and mod. NSRL models, with the same point of inflection for each pair, are represented. Parameters δ and σ have been chosen from eq. (25). The

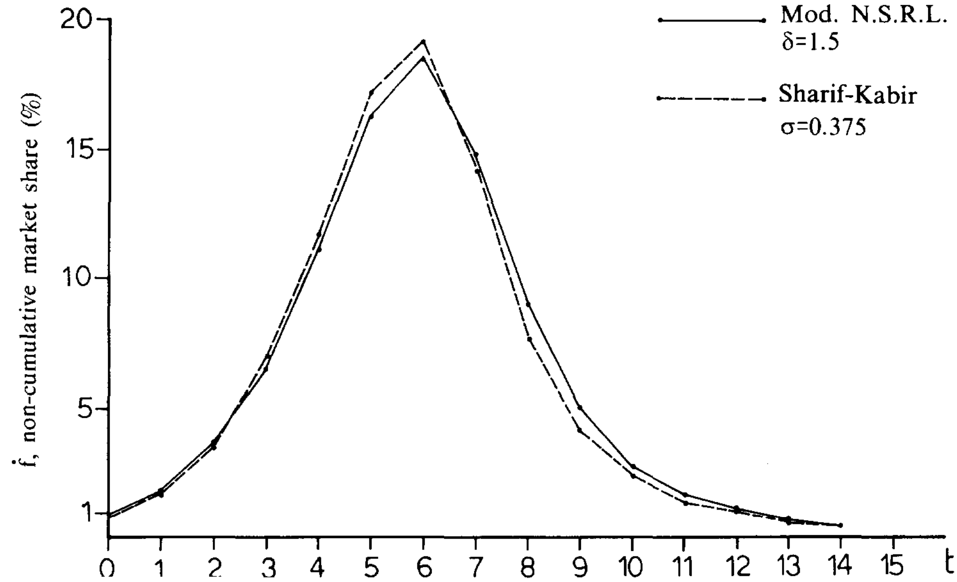

Fig. 6. Comparison between Sharif-Kabir and Modified NSRL models. Initial period market share is 1% and b = 1. Point of inflection at f = 0.40F.

curves, representing the compared models, approach each other very well, as expected by the preceding theory.

The Generalized Rational Model II (GRM II)

This model arises from the NSRL model with the method which was applied for the formulation of the GRM I model. Thus differential equation (11) representing the NSRL model, is expressed

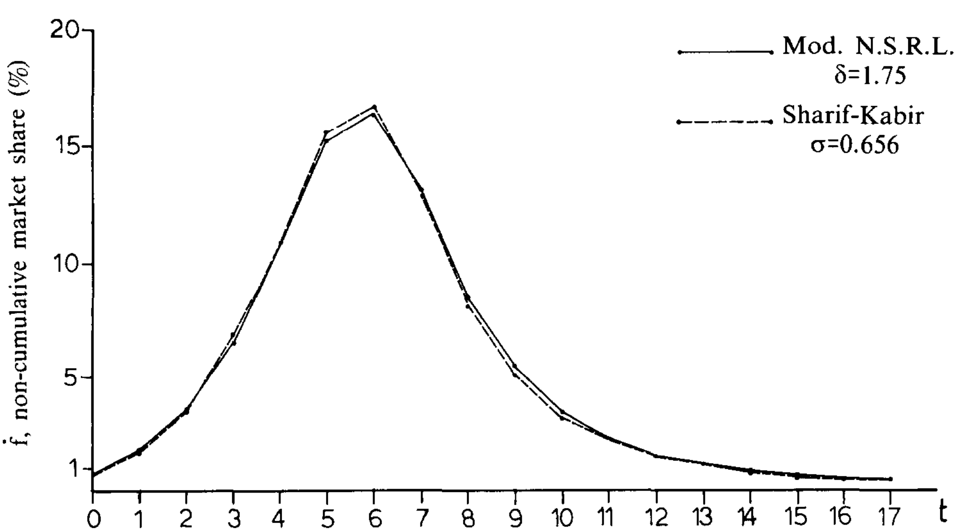

Fig. 7. Comparison between Sharif-Kabir and Modified NSRL models. Initial period market share is 1% and b = 1. Point of inflection at f = 0.36F.

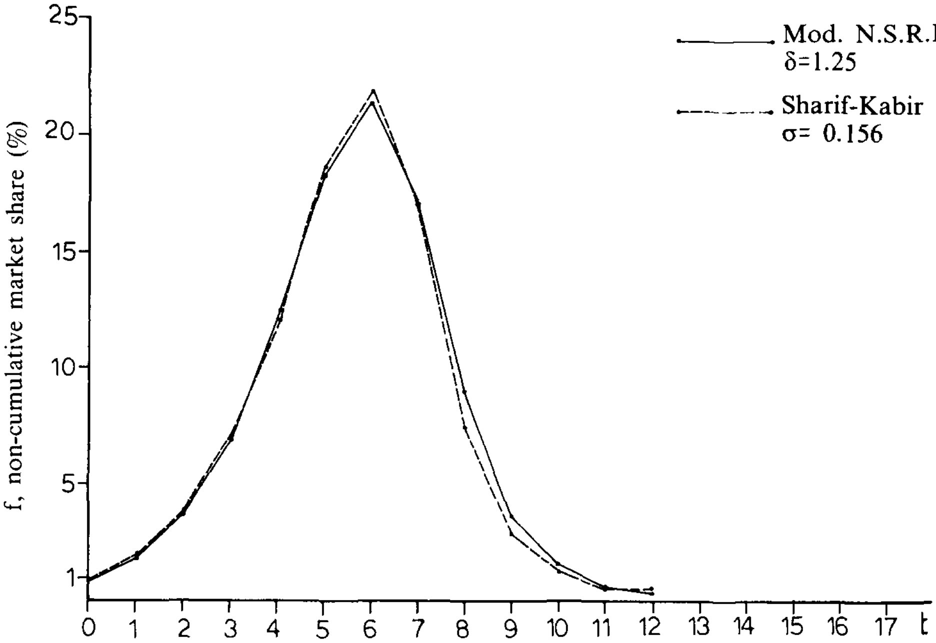

Fig. 8. Comparison between Sharif-Kabir and Modified NSRL models. Initial period market share is 1% and b = 1. Point of inflection at f leq 0.44F.

$$\dot{f} = b\left(\frac{f}{F}\right)^{n-\varepsilon}(F - f), \tag{29}$$

where $n - \varepsilon = \delta$ if $0 \leq \varepsilon < 1$, n is an integer varying from 1 to ∞ when δ varies from 0 to ∞, and $\delta < n \leq \delta + 1$.

From eq. (29) follows

$$\dot{f} = b\,\frac{\left(\frac{f}{F}\right)^{n}}{\left(\frac{f}{F}\right)^{\varepsilon}}(F - f). \tag{30}$$

Substituting the denominator in the right side of eq. (30) by the approximation provided in Appendix B (eq. (B-5), eq. (30) yields:

$$\dot{f} = b\,\frac{\left(\frac{f}{F}\right)^{n}}{1 - \varepsilon + \varepsilon\frac{f}{F}}(F - f). \tag{31}$$

Substituting $(1 - \sigma)$ for ε in eq. (31) and rearranging gives the differential equation of the new model, defined as the "Generalized Rational Model II" (GRM II):

$$\dot{f} = b \frac{f^n}{F^{n-1}[\sigma F + (1 - \sigma)f]} (F - f), \tag{32}$$

where $n = 1,2, \ldots$, and $0 \leq \sigma \leq 1$.

This model covers the same region as the NSRL model illustrated in Figure 4. For $n = 1$ and $\sigma = 0$, eq. (32) becomes the Coleman model, while for $n = 1$ and $\sigma = 1$ or for $n = 2$ and $\sigma = 0$, it becomes the Blackman/Fisher-Pry model. For $n = 2$ and $\sigma = 1$ or for $n = 3$ and $\sigma = 0$, eq. (13) is the result.

The solution of differential equation (32) of the GRM II model is provided by eq. (C-10) in Appendix C:

$$\ln f - \ln(F - f) - \frac{F}{f} - \ldots - \frac{F^{n-2}}{(n-2)f^{n-2}} - \sigma \frac{F^{n-1}}{(n-1)f^{n-1}} = c + bt, \tag{33}$$

where $n = 2,3, \ldots$. When $n = 1$ the solution is given by eq. (C − 12) in Appendix C:

$$\sigma \ln f - \ln(F - f) = c + bt. \tag{34}$$

Note that c is a constant, the value of which is obtained by setting $t = 0$.

The GRM II model is a mirror image of the GRM I. Most equations involving GRM II can be provided from the GRM I's equations by setting $f = F - f$. So, the point of inflection of the GRM II model, after transformation of eq. (22), is given by

$$f = F\left[1 - \frac{(n + 1) - \sqrt{(n + 1)^2 - 4n(1 - \sigma)}}{2n(1 - \sigma)}\right]. \tag{35}$$

The GRM II and NSRL models have the same f in the point of inflection when

$$\sigma = \frac{\delta(1 + \delta - n)}{n}. \tag{36}$$

This last equation is identical to eq. (23), which is valid for parameters σ, n and δ of the GRM I and mod. NSRL models.

By setting $n = 1$ in eq. (32), the GRM II_1 model is provided with the differential equation:

$$\dot{f} = b \frac{f}{\sigma F + (1 - \sigma)f} (F - f). \tag{37}$$

A point of inflection is given by:

$$f = F \frac{- \sigma + \sqrt{\sigma}}{1 - \sigma}. \tag{38}$$

This model is of great importance, representing a mixed diffusion process with both internal interaction (imitation) and external influence (a Coleman process). Parameter σ

provides a measure of the proportion of those two general forces which mainly influence innovation diffusion.

By setting $n = 2$ in eq. (32), the GRM II_2 model is provided with the differential equation

$$\dot{f} = b \frac{f^2}{F[\sigma F + (1 - \sigma)f]} (F - f), \tag{39}$$

which has the solution

$$\ln f - \ln (F - f) - \sigma \frac{F}{f} = c + bt. \tag{40}$$

The point of inflection of the GRM II_2 model is given by the equation

$$f = F \frac{1 - 4\sigma + \sqrt{1 + 8\sigma}}{4(1 - \sigma)}. \tag{41}$$

When $\sigma = 1$, eq. (39) provides a model, symmetrical with the Floyd model, which is identical to the model expressed by eq. (13) and provided by the NSRL model when $\delta = 2$.

Comparing the GRM II and the NSRL Models

To compare the Generalized Rational Model II (GRM II) with the NSRL model, the same data are used as for the NSRL in [3].

A nonlinear regression analysis algorithm is used to define the parameters of the model. Table 1 summarizes the parameter estimates, and Table 2 shows the computed points of inflection and the mean squared error for the GRM II and the NSRL models. The results show the close relationship between the two models. The mean squared error for the CT head scanner and mammography in the NSRL are 0.98 and 1.04 of the values estimated for the GRM II, respectively. For ultra-sound the NSRL model gives better results with a ratio of 0.90. For the CT body scanner the GRM II fits the data very well with a ratio of 2.68 and mean squared error 1.52.

The CT head scanner and the CT body scanner are described by eq. (37), parameter n of the GRM II_1 model being equal to 1. Figures 9 and 10 show the approximation between the GRM II_1 and the NSRL models. Ultrasound and Mammography have parameter $n = 2$, so they are described by eq. (39). Figures 11 and 12, like Figures 9 and 10, show the similar behavior of the two models.

Although, in the two last innovations, σ approaches zero significantly (0.0478 and

TABLE 1.
Parameter Estimates for Four Medical Technological Innovations

	NSRL			GRM II			
Product	b	F	δ	n	b	F	σ
CT Head Scanner	0.9645	0.56	0.6644	1	0.6500	0.58	0.1866
CT Body Scanner	1.3996	0.47	0.7899	1	0.9547	0.50	0.3129
Ultrasound	0.7535	0.80	1.2191	2	0.6901	0.80	0.0478
Mammography	0.8735	0.56	1.1215	2	0.8349	0.55	0.0184

TABLE 2.
Point of Inflection and Mean Squared Error in NSRL and GRM II Models

Product	Point of Inflection NSRL	Point of Inflection GRM II	Mean Squared Error NSRL	Mean Squared Error GRM II	Ratio NSRL/GRM II
CT Head Scanner	0.40F	0.30F	10.48	10.64	0.98
CT Body Scanner	0.44F	0.36F	4.08	1.52	2.68
Ultrasound	0.55F	0.52F	6.28	6.99	0.90
Mammography	0.53F	0.51F	3.13	3.02	1.04

0.0184, as indicated in Table 1), the estimated mean squared error is 7.27 and 3.11, respectively, instead of 12.42 and 5.23 as given by the Blackman/Fisher-Pry model [3] provided by eq. (39) when $\sigma = 0$.

Testing the GRM I and Mod. NSRL Models in Specific Processes

To test these models, data for the diffusion of Oxygen Steel Technology in West Germany and in Spain are examined. These data [11] provide time-series from 1957 to 1978 for West Germany and from 1963 to 1980 for Spain. In them, f expresses the percentage of steel produced by the oxygen steel process. Therefore, F is less than or equal to 100.

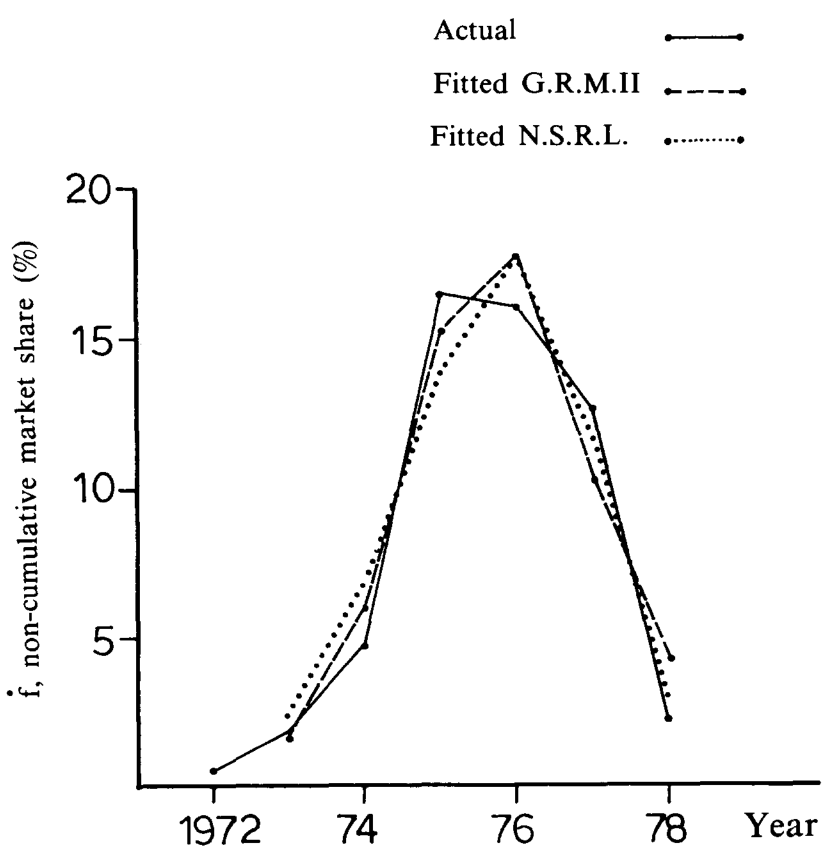

Fig. 9. Actual and fitted GRM II and NSRL noncumulative market share for CT head scanner.

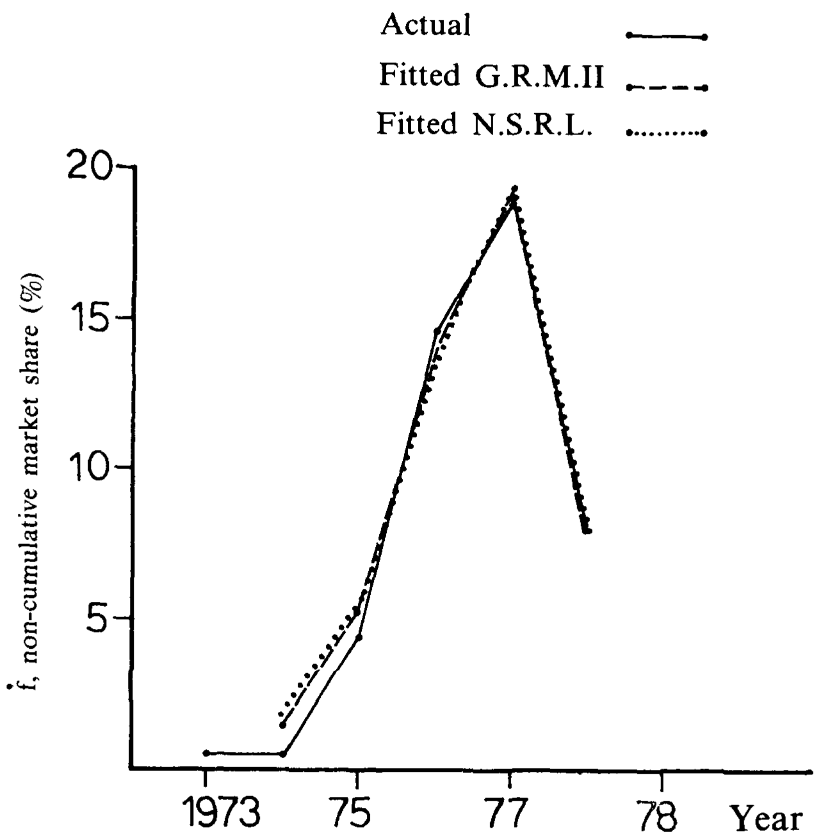

Fig. 10. Actual and fitted GRM II and NSRL noncumulative market share for CT body scanner.

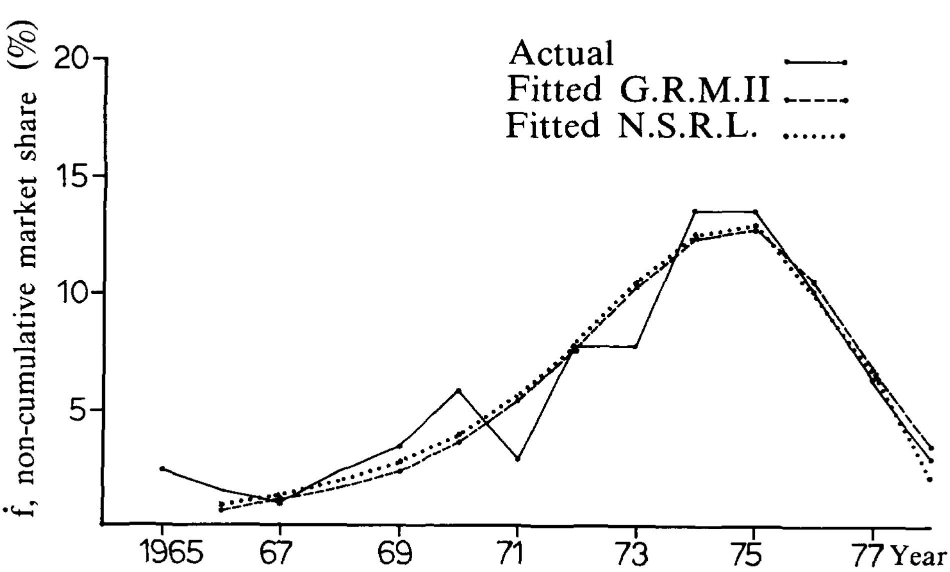

Fig. 11. Actual and fitted GRM II and NSRL noncumulative market share for ultrasound.

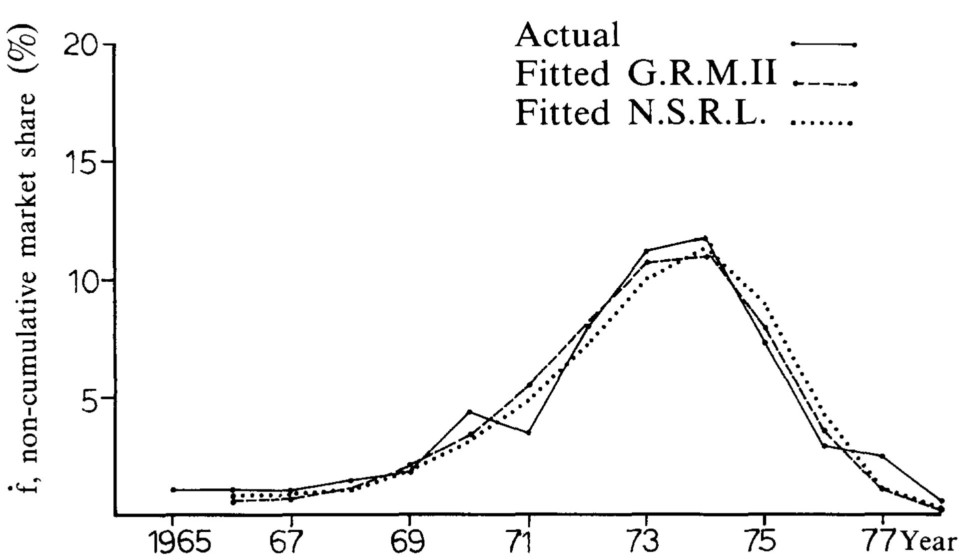

Fig. 12. Actual and fitted GRM II and NSRL noncumulative market share for mammography.

By using a nonlinear regression analysis algorithm, parameters of the GRM I, mod. NSRL, Blackman/Fisher-Pry, Coleman, Floyd and Exponential models are estimated. Parameter estimates are summarized in Tables 3 and 4 where f_0 expresses the market share covered by the new product at time $t = 0$. The two tables also provide the Mean Squared Error (MSE), the point of inflection, and the Coefficient of Determination (also known as Coefficient of Correlation), corrected for degrees of freedom $\bar{R}^2$ [9].

The base for comparisons between the models is MSE. In both cases the GRM I provides a better fit than the other models. The diffusion of the steel, produced by the oxygen process in West Germany (Table 3), follows the GRM I_2/Sharif-Kabir model, while the mod. NSRL model approximates it very well. The ratio of the MSE NSRL Model to the MSE GRM I_2 Model is 1.0290, though both models provide very close inflection points at 34.0770 and 34.2924 for the GRM I_2 and mod. NSRL models, respectively.

For Spain, the GRM I_1 model seems to be the most appropriate, providing the smallest MSE (5.7499) and the biggest $\bar{R}^2$ (0.9759) among all the models illustrated in Table 4. The ratio of the NSRL to the GRM I_1 for MSE is 1.0658, indicating the approximation between the two models (see Table 4).

Testing the GRM II and NSRL Models

Time-series data for the world diffusion of Oxygen Steel Technology (1960–1978) and for the diffusion of the same technology in France (1960–1980) [11] are examined. As indicated in Table 5, the GRM II_1 model fits the data very well for the world diffusion, giving a value of 1.4750 for MSE, which is the 0.8093 of the value estimated for the NSRL model.

Parameters $n = 1$ and $\sigma = 0.1470$ indicate that the World diffusion process is influenced by both external influence (Coleman) and imitation (Blackman/Fisher-Pry). The inflection point is very low, at only 18.5753 percent of the market share, a level that was covered in 1965, indicating that a decline in the production of high-quality steel by the oxygen process, in the world steel industry, began almost ten years before the big

TABLE 3.
Parameter Estimates, $\bar{R}^2$, and MSE for the Diffusion of Oxygen Steel Technology in West Germany from 1957 to 1978

	GRM I	mod. NSRL	Blackman/ Fisher-Pry	Coleman	Floyd	Exponential
b	0.4915	0.4923	0.4508	0.0501	0.6049	0.1091
f_0	0.8741	0.8794	1.0737	0.0001	0.5391	10.4707
F	79.1567	75.5673	73.0192	100	89.1715	—
n	2	—	—	—	—	—
σ	0.2134	—	—	—	—	—
δ	—	1.2037	—	—	—	—
$\bar{R}^2$	0.9984	0.9984	0.9980	0.8401	0.9976	0.8446
MSE	1.0456	1.0759	1.3721	124.5098	1.6972	114.9375
Inflection Point	0.4305F = 34.0770	0.4538F = 34.2924	F/2 = 36.5096	—	F/3 = 29.7238	—

Data from [11]

crisis in the world steel market. This last fact demonstrates the possibility for forecasting future crises when sufficient data are provided and appropriate models are used.

For France, the NSRL model fits the data very well with a MSE of 8.9979, which is 0.9957 percent of the value estimated for the GRM II_2 model. The two models approximate very well, showing a small divergence from the Blackman/Fisher-Pry model, as indicated in Table 6. The whole diffusion process in France, mainly influenced by imitation, shows a relatively higher inflection point (45.0752 percent) than that of West Germany, Spain and the world.

Conclusions

The GRM I and II models, proposed in this paper, in accordance with the NSRL model proposed by Easingwood, Mahajan and Muller, and the mod. NSRL, also introduced hereby, seem to have a good ability to forecast innovation diffusion. These models cover the existing binomial models—Coleman, Exponential, Blackman/Fisher-Pry, Floyd

TABLE 4.
Parameter Estimates, $\bar{R}^2$, and MSE for the Diffusion of Oxygen Steel Technology in Spain from 1963 to 1980

	GRM I	mod. NSRL	Blackman/ Fisher-Pry	Coleman	Floyd	Exponential
b	0.4053	0.5101	0.5437	0.1285	0.8302	0.0702
f_0	5.5052	4.6625	4.4274	0.0001	3.1096	19.6007
F	51.3129	50.8129	51.7641	60.8517	59.0341	—
n	1	—	—	—	—	—
σ	0.4748	—	—	—	—	—
δ	—	0.8853	—	—	—	—
$\bar{R}^2$	0.9759	0.9743	0.9746	0.9107	0.9521	0.6134
MSE	5.7499	6.1281	6.5249	22.9101	12.2751	106.2401
Inflection Point	0.5920F = 30.3772	0.5304F = 26.9512	F/2 = 25.8821	—	F/3 = 19.6780	—

Data from [11]

TABLE 5.
Parameter Estimates, $\bar{R}^2$ and MSE for the World Diffusion of Oxygen Steel Technology from 1960 to 1978

	GRM II	NSRL	Blackman/ Fisher-Pry	Coleman	Floyd	Exponential
b	0.1419	0.1878	0.3641	0.0488	0.5018	0.0962
f_0	1.8998	1.4400	4.4770	0.0001	3.2292	13.2037
F	67.0345	63.9496	56.9694	100	70.3279	—
n	1	—	—	—	—	—
σ	0.1470	—	—	—	—	—
δ	—	0.4756	—	—	—	—
$\bar{R}^2$	0.9949	0.9941	0.9863	0.9782	0.9937	0.8583
MSE	1.4750	1.8226	4.2628	7.7039	1.9665	47.0862
Inflection Point	0.2771F = 18.5753	0.3223F = 20.6110	F/2 = 28.4847	—	F/3 = 23.4426	—

Data from [11]

and Sharif-Kabir—except for the Generalized model proposed by Mahajan and Schoeman, which is oriented above the line representing the Coleman model (see Figure 2).

Most effort in this work has centered on proving the approximation (usually close enough) between the GRM I and mod. NSRL models and the GRM II and NSRL models, in order to introduce alternate general models that can be useful in different areas. The GRM I and II models seem to be more useful in applications where differential equation solution are required, while the NSRL and mod. NSRL models, providing simple equations for inflection points, can be more appropriate for comparisons with other models.

All general models analyzed here exhibit behavior more complicated than imitation, which certainly remains always a main force influencing innovation diffusion; the Blackman/Fisher-Pry model representing diffusion by imitation remains a special case among all these models.

Work in this area must be carried on. For many reasons, innovation diffusion—especially the diffusion of new technologies—will probably cause a series of actions which may lead to changes in social behavior and in economical and other factors. Then,

TABLE 6.
Parameter Estimates, $\bar{R}^2$, and MSE for the Diffusion of Oxygen Steel Technology in France from 1960 to 1980

	GRM II	NSRL	Blackman/ Fisher-Pry	Coleman	Floyd	Exponential
b	0.3357	0.3549	0.3089	0.0511	0.4853	0.1259
f_0	3.4329	3.2265	2.6529	0.0001	1.0220	8.9323
F	86.6593	86.0049	88.9385	100	100	—
n	2	—	—	—	—	—
σ	0.0225	—	—	—	—	—
δ	—	1.1012	—	—	—	—
$\bar{R}^2$	0.9857	0.9866	0.9864	0.8169	0.9763	0.9267
MSE	9.0365	8.9979	9.1344	137.7615	15.9627	52.2260
Inflection Point	0.5106F = 44.2482	0.5241F = 45.0752	F/2 = 44.4693	—	F/3 = 50	—

Data from [11]

the characteristics and the dynamics of many changes may be represented by parameters such as n, σ and δ.

Appendix A: Approximations Dealing with the GRM I and Mod. NSRL Models

The approximation of

$$\left(1 - \frac{f}{F}\right)^{\varepsilon},$$

where $0 \leqslant \varepsilon \leqslant 1$ and $0 < (f/F) < 1$, with a simpler function, is derived as:

$$\left(1 - \frac{f}{F}\right)^{\varepsilon} = e^{\varepsilon \ln\left(1 - \frac{f}{F}\right)} \tag{A-1}$$

Expanding the natural logarithm in a series

$$\left(1 - \frac{f}{F}\right)^{\varepsilon} = e^{\varepsilon\left(-\frac{f}{F} - \frac{1}{2}\frac{f^2}{F^2} - \cdots\right)} \tag{A-2}$$

can be approximated by

$$\left(1 - \frac{f}{F}\right)^{\varepsilon} = e^{-\varepsilon\frac{f}{F}}. \tag{A-3}$$

The approximation is best in the first stages of the diffusion process when $f/F \ll 1$.

Expanding the right side of eq. (A-3) in a series, it gives

$$\left(1 - \frac{f}{F}\right)^{\varepsilon} = 1 - \varepsilon\frac{f}{F} + \frac{1}{2!}\left(\varepsilon\frac{f}{F}\right)^2 - \ldots, \tag{A-4}$$

approximated by

$$\left(1 - \frac{f}{F}\right)^{\varepsilon} = 1 - \varepsilon\frac{f}{F}. \tag{A-5}$$

Best approximation is obtained when

$$\varepsilon\frac{f}{F} \ll 1. \tag{A-6}$$

Appendix B. Approximations Dealing with the GRM II and NSRL Models

The approximation of

$$\left(\frac{f}{F}\right)^{\varepsilon},$$

where $0 \leq \varepsilon \leq 1$ and $0 < (f/F) < 1$, with a simpler function is derived as:

$$\left(\frac{f}{F}\right)^{\varepsilon} = e^{\varepsilon \ln\left(\frac{f}{F}\right)} = e^{\varepsilon \ln\left[1-\left(1-\frac{f}{F}\right)\right]}, \tag{B-1}$$

where $0 < [1 - (f/F)] < 1$.

Expanding the natural logarithm in a series

$$\left(\frac{f}{F}\right)^{\varepsilon} = e^{\varepsilon\left[-\left(1-\frac{f}{F}\right)-\frac{1}{2}\left(1-\frac{f}{F}\right)^{2}-\cdots\right]} \tag{B-2}$$

can be approximated by

$$\left(\frac{f}{F}\right)^{\varepsilon} = e^{-\varepsilon\left(1-\frac{f}{F}\right)}. \tag{B-3}$$

The approximation is best in the last stages of the diffusion process when $(1 - f/F) \ll 1$.

Expanding the right side of eq. (B-3) in a series, it gives

$$\left(\frac{f}{F}\right)^{\varepsilon} = 1 - \varepsilon\left(1 - \frac{f}{F}\right) + \frac{1}{2!}\varepsilon^{2}\left(1 - \frac{f}{F}\right)^{2} - \cdots, \tag{B-4}$$

approximated by

$$\left(\frac{f}{F}\right)^{\varepsilon} = 1 - \varepsilon\left(1 - \frac{f}{F}\right)$$

or

$$\left(\frac{f}{F}\right)^{\varepsilon} = 1 - \varepsilon + \varepsilon\frac{f}{F}. \tag{B-5}$$

Best approximation is obtained when

$$\varepsilon\left(1 - \frac{f}{F}\right) \ll 1. \tag{B-6}$$

Appendix C. Solution of the Differential Equations of the GRM I and GRM II Models.

The differential equation (eq. (19)) of the GRM I model can be expressed as:

$$\frac{F^{n-1}[F - (1 - \sigma)f]}{f(F - f)^{n}}\, df = b\,dt \tag{C-1}$$

or

$$\frac{F^{n-1}(F - f) + \sigma F^{n-1} f}{f(F - f)^n} df = bdt,$$

which gives

$$\left[\frac{F^{n-1}}{f(F - f)^{n-1}} + \sigma \frac{F^{n-1}}{(F - f)^n}\right] df = bdt. \qquad \text{(C-2)}$$

expanding the first term of the left side of (C-2) in a series

$$\left[\frac{1}{f} + \frac{1}{F - f} + \frac{F}{(F - f)^2} + \cdots + \frac{F^{n-2}}{(F - f)^{n-1}} + \frac{\sigma F^{n-1}}{(F - f)^n}\right] df = bdt, \qquad \text{(C-3)}$$

the integration of (C-3) yields

$$\ln f - \ln (F - f) + \frac{F}{F - f} + \cdots + \frac{F^{n-2}}{(n - 2)(F - f)^{n-2}} + \sigma \frac{F^{n-1}}{(n - 1)(F - f)^{n-1}} = c + bt \qquad \text{(C-4)}$$

where c is a constant and $n = 2,3,\ldots$. When $n = 1$, eq. (C-2) yields

$$\left(\frac{1}{f} + \frac{\sigma}{F - f}\right) df = bdt, \qquad \text{(C-5)}$$

and by integration

$$\ln f - \sigma \ln (F - f) = c + bt. \qquad \text{(C-6)}$$

The differential equation (eq. (32)), expressing the GRM II model, can be written

$$\frac{F^{n-1}[\sigma F + (1 - \sigma)f]}{f^n(F - f)} df = bdt \qquad \text{(C-7)}$$

or

$$\frac{F^{n-1}[\sigma(F - f) + f]}{f^n(F - f)} df = bdt,$$

which gives

$$\left[\frac{F^{n-1}}{f^{n-1}(F - f)} + \sigma \frac{F^{n-1}}{f^n}\right] df = bdt. \qquad \text{(C-8)}$$

Expanding the first term of the left side in a series

$$\left[\frac{1}{(F - f)} + \frac{1}{f} + \frac{F}{f^2} + \cdots + \frac{F^{n-2}}{f^{n-1}} + \sigma \frac{F^{n-1}}{f^n}\right] df = bdt \qquad (C\text{-}9)$$

and integrating eq. (C-9), the general equation of GRM II is

$$\ln f - \ln (F - f) - \frac{F}{f} - \cdots - \frac{F^{n-2}}{(n - 2)f^{n-2}} - \sigma \frac{F^{n-1}}{(n-1)f^{n-1}} = c + bt, \qquad (C\text{-}10)$$

where c is a constant, and $n = 2,3, \ldots$. When $n = 1$, eq. (C-8) yields

$$\left[\frac{1}{F - f} + \frac{\sigma}{f}\right] df = bdt, \qquad (C\text{-}11)$$

and integrating the last equation

$$\sigma \ln f - \ln(F - f) = c + bt. \qquad (C\text{-}12)$$

The author wishes to acknowledge some valuable comments made on an earlier version of this paper by Professor N. Koumoutsos of the Athens National Technical University, Department of Chemical Engineering, Processes and Systems Engineering. He is also grateful for the useful suggestions of an anonymous referee.

References

1. Blackman, A. W. Jr. A mathematical model for trend forecasts, *Technol. Forecast. Soc. Change* 3, 441–452, (1972).
2. Bretchneider, S. I., and Mahajan V. Adaptive technological substitution models, *Technol. Forecast. Soc. Change* 18, 129–139 (1980).
3. Easingwood, C., Mahajan, V., and Muller, E. A nonsymmetric responding logistic model for forecasting technological substitution, *Technol. Forecast. Soc. Change* 20, 199–213 (1981).
4. Fisher, J. C., and Pry, R. H. A simple substitution model of technological change, *Technol. Forecast. Soc. Change* 3, 75–88 (1971).
5. Haynes, K. E., Mahajan, V., and White, G. M. Innovation diffusion: A deterministic model of space-time integration with physical analog, *Socio-Econ. Plan. Sci., vol. II*, 25–29 (1977).
6. Mahajan V., Haynes, K. E., and Bal Kumar, K. C. Modeling the diffusion of public policy innovations among the U.S. states, *Socio-Econ. Plan. Sci., vol. II*, 259–263 (1977).
7. Mahajan, V., and Muller, E. Innovation diffusion and new product growth models in marketing, *J. Marketing* 43, 55–68 (1979).
8. Mahajan, V., and Schoeman, M. E. F. Generalized model for the time pattern of the diffusion process, *IEEE Trans. Eng. Management* 1, 12–18 (1977).
9. Makridakis, S., Wheelwright, S. C., and McGee, V.E. *Forecasting Methods and Applications*, Wiley, New York, pp. 260, 659 (1983).
10. Mansfield, E. Technical change and the rate of imitation, *Econometrica* 29, 741–766 (1961).
11. Poznanski, K. International diffusion of steel technologies, time-lag and the speed of diffusion. *Technol. Forecast. Soc. Change* 23, 305–323 (1983).
12. Saaty, T. L., and Bram, J. Nonlinear Mathematics, Dover, New York, p. 255 (1981).
13. Sharif, M. N., and Kabir, C. A. Generalized model for forecasting technological substitution, *Technol. Forecast. Soc. Change* 8, 353–364 (1976).
14. Sharif, M. N., and Kabir, C. System dynamics modeling for forecasting multilevel technological substitution, *Technol. Forecast. Soc. Change* 9, 89–112 (1976).

15. Sharif, M. N., and Ramanathan, K. Binomial innovation diffusion models with dynamic potential adopter population, *Technol. Forecast. Soc. Change* 20, 63–87 (1981).

Revised 2 July 1984